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LETTER TO THE EDITOR

Generating function for a multiplicative noise with group analysis

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Abstract. We find the differential equation for the generating function of a multiplicative stochastic process and we apply to it the group analysis. We give the general form of the Lie generators and find the conditions for the existence of similarity solutions. Two classes of similarity solutions are presented and the analytic expression of the generating function is given.

The behaviour of nonlinear dynamical systems subjected to multiplicative noise has been the subject of increasing interest in recent years [1–3], because of the peculiar role of this type of noise. In the framework of nonlinear systems the presence of multiplicative fluctuations, due to an irregular influence imposed on the system from the environment to which it is coupled, can determine a behaviour qualitatively different from that which arises from an additive noise [2]. A common way of dealing with the random processes defined by a multiplicative stochastic differential equation (SDE) is to look for a solution of the corresponding Fokker–Planck equation (FP), which is statistically equivalent to the given SDE [4].

Recently considerable attention has been paid to the determination of the generalized symmetries and of similarity solutions of the one-dimensional Fokker–Planck equation [5–7].

In this letter, using the statistical properties of the Wiener process, we find the differential equation for the generating function (DEGF) of a multiplicative stochastic process and we investigate its invariance under continuous groups of transformations. Particularly, we get the conditions for the existence of similarity solutions and an exact analytic expression for the generating function using the constraints of analyticity and of the initial condition. As is well known the generating function contains all the statistical information on the system and its determination is directly relevant to measurable quantities. We want to point out that this approach can be considered alternative to the direct investigation of the FP equation and can be very useful when the general solution of this equation is mathematically involved. Our starting point is the following SDE:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=[b+\xi(t)]x+a \qquad (1)$$

where a, b and $\xi(t)$ stand, respectively, for an ignition parameter, a constant parameter and a white noise source satisfying

$$\langle\xi(t)\rangle=0 \qquad \langle\xi(t)\xi(t')\rangle=\varepsilon\delta(t-t')$$

and ε is the strength of the multiplicative noise. The equivalent Ito form of equation (1) is

$$dx=[(b+\varepsilon/2)+\sqrt{\varepsilon}\,dW]x+a\,dt \tag{2}$$

where $W(t)$ is the Wiener process with the usual statistical properties.

Integrating (2) and changing variable [8] we get the formal solution as a functional of the Wiener process:

$$x(t)=a\int_0^t e^{bt'+\sqrt{\varepsilon}W(t')}\,dt'. \tag{3}$$

Conversely using the Ito formula of the stochastic calculus it is easy to show that differentiating equation (3) we obtain (2). To get the DEGF we use the differential equation of the moments (DEM) which can be easily obtained, as is well known, from the Fokker–Planck equation corresponding to our SDE (2):

$$\frac{\partial P(x,t)}{\partial t}=-\frac{\partial}{\partial x}[(b+\tfrac{3}{2})x+a]P(x,t)+\frac{\varepsilon}{2}\frac{\partial^2}{\partial x^2}[x^2P(x,t)]. \tag{4}$$

We present here an alternative derivation of the DEM based on the properties of the Winer process. In fact the increments of the Wiener process are statistically independent, therefore we can discretize the integral of equation (3) writing

$$\begin{aligned}x(t)&=\lim_{\Delta t\to 0} a\Delta t\left[1+\sum_{n=1}^{\infty} e^{nb\Delta t}\prod_{l=1}^{n} e^{\Delta w_l}\right]\\&=\lim_{\Delta t\to 0}[a\Delta t(1+yz)]=\lim_{\substack{\Delta t\to 0\\ N\to\infty}} a\Delta t\chi_n\end{aligned} \tag{5}$$

where

$$\begin{aligned}y(t)&=\exp[b\Delta t+\sqrt{\varepsilon}\,\Delta w_1(t)]\\ z(t)&=1+\exp[b\Delta t+\sqrt{\varepsilon}\,\Delta w_2(t)]+\exp[2b\Delta t+\sqrt{\varepsilon}\,\Delta w_3(t)]+\ldots\end{aligned} \tag{6}$$

and

$$x_N(t)=1+\sum_{n=1}^{N-1} e^{nb\Delta t}\prod_{l=1}^{n} e^{\sqrt{\varepsilon}\Delta w_l(t)}=1+yx_{N-1}. \tag{6a}$$

$y(t)$ and $z(t)$ are independent processes, and $z(t)$ has the same probability distribution of the process $x(t)$. We now write equation (5) as a recurrence relation by using (6a) and taking the mean values of both sides:

$$\langle(x_N-1)^k\rangle=\langle y^k\rangle\langle x_{n-1}^k\rangle$$

but

$$\langle(x_N-1)^k\rangle=\langle x_N^k\rangle+\sum_{n=1}^{\infty}\binom{k}{n}(-1)^n\langle x_N^{k-n}\rangle$$

therefore

$$\langle x_N^k\rangle_t+\sum_{n=1}^{\infty}\binom{k}{n}(-1)^n\langle x_N^{k-n}\rangle_t=\langle y^k\rangle\langle x_{N-1})^k\rangle_t. \tag{7}$$

Now multiplying (7) by $(a\Delta t)^k$, using the identity

$$\langle x_{N-1}^k\rangle_t=\langle x_N^k\rangle_{t-\Delta t}$$

and the well known Gaussian property of the Wiener process

$$\langle y^k\rangle=\langle(\exp[b\Delta t+\Delta w])^k\rangle=\exp[bk\Delta t+k^2/2\Delta t]\simeq 1+\left[bk+\varepsilon\frac{k^2}{2}\right]\Delta t+\ldots$$

retaining terms of order Δt, we obtain the differential equation for the moments

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x^n\rangle_t=\left(bn+n^2\frac{\varepsilon}{2}\right)\langle x^n\rangle_t+na\langle x^{n-1}\rangle_t \qquad n=1,2,\ldots \tag{8}$$

Therefore the differential equation for the generating function (DEGF) can be easily obtained as

$$\frac{\partial G_x(\lambda,t)}{\partial t}=-\lambda aG_x+\left(b+\frac{\varepsilon}{2}\right)\lambda\frac{\partial G_x}{\partial\lambda}+\frac{\varepsilon}{2}\lambda^2\frac{\partial^2 G_x}{\partial\lambda^2}. \tag{9}$$

Following [8] the group analysis of equation (9) is performed through the one-parameter Lie group of transformations:

$$\begin{aligned}\lambda'&=\lambda+\varepsilon\Lambda(\lambda,t,G_x)+o(\varepsilon^2)\\ t'&=t+\varepsilon T(\lambda,t,G_x)+o(\varepsilon^2)\\ G_x'&=G_x+\varepsilon G(\lambda,t,G_x)+o(\varepsilon^2)\end{aligned} \tag{10}$$

where ε is a continuous parameter and the group infinitesimals (Λ, T, G) must satisfy the following determining equations (the subscript meaning differentiation):

$$\Lambda_g=T_\lambda=T_g=G_{gg}=0 \tag{11a}$$

$$-2\Lambda-\lambda T_t+2\lambda\Lambda=0 \tag{11b}$$

$$\left(b+\frac{\varepsilon}{2}\right)\Lambda-\left(b+\frac{\varepsilon}{2}\right)\lambda\Lambda_\lambda-\varepsilon\lambda^2G_{g\lambda}+\frac{\varepsilon}{2}\lambda^2\Lambda_{\lambda\lambda}-\Lambda_t=0 \tag{11c}$$

$$a\lambda G-ag\Lambda-\left(\frac{b+\varepsilon}{2}\right)\lambda G_\lambda-\frac{\varepsilon}{2}\lambda^2G_{\lambda\lambda}-a\lambda gG_g+2a\lambda g\Lambda_\lambda+G_t=0 \tag{11d}$$

where $g=G$. Integration of (11a) and (11b) leads to the following form of the infinitesimal generators:

$$\begin{aligned}\Lambda&=\frac{T_t}{2}\lambda\ln\lambda+C_1(t)\lambda\\ T&=T_t\\ G&=A(\lambda,t)g+B(\lambda,t).\end{aligned} \tag{12}$$

Without loss of generality we can set $B(\lambda, t) = 0$. Substituting (12) into (11*c*) and (11*d*) we obtain the conditions for the existence of similarity solutions:

$$\frac{\varepsilon}{2}\lambda^2 A_{\lambda\lambda} + \left(b + \frac{\varepsilon}{2}\right)\lambda A_\lambda - a\Lambda - a\lambda T_t - A_t = 0 \tag{13}$$

$$\frac{\varepsilon}{2}\lambda^2(2A_\lambda - \Lambda_{\lambda\lambda}) + \left(b + \frac{\varepsilon}{2}\right)\lambda(T_t - \Lambda_t) + \left(b + \frac{\varepsilon}{2}\right)\Lambda + \Lambda_t = 0. \tag{14}$$

Now we can easily deal with two particular cases and discuss briefly the corresponding solution.

Case 1
$A = 1$, $T = \text{constant} = 1/\alpha$. The infinitesimal are

$$\Lambda = 0 \qquad T = 1/\alpha \qquad G = g$$

and the invariants of the subgroup of transformations are:

$$\mu_1 = \lambda \qquad \mu_2 = g\,\mathrm{e}^{-\alpha t}.$$

We note that inspecting more closely the conditions (13) and (14), one easily sees that the only solution allowed is that obtained with $C_1(t) = 0$ and $T_t = 0$, which leads to the 'trivial' solution, corresponding to a trivial symmetry. The similarity solution is:

$$g(\lambda, t) = C_1\,\mathrm{e}^{\alpha t} f(\lambda) \tag{15}$$

and $f(\lambda)$ satisfies a differential equation leading to the Bessel differential equation through the product transformation [9]

$$f(\lambda) = \lambda^{1/2 - a/2b}\,\mathrm{e}^{\pi/2\nu i}\, I_\nu\left(\frac{2\lambda a}{\varepsilon}\right)^{1/2} \tag{16}$$

where:

$$\nu = (b^2/\varepsilon^2 + 2\alpha/\varepsilon)^{1/2} \tag{16'}$$

and I_ν is the modified Bessel function. To satisfy the constraint of analyticity for the generating function when $\lambda \to 0$, we reject the second linearly independent solution K_ν and discretize the parameter ν as:

$$\frac{\nu_k}{2} - \frac{b}{\varepsilon} = k \qquad k = 0, 1, 2, \ldots. \tag{17}$$

Then, because of the linearity of (9), we can construct an exact analytic expression for the generating function with the expansion

$$\begin{aligned} G_x(\lambda, t) &= \sum_{k=0}^{\infty} C_k\,\mathrm{e}^{\alpha_k t}\lambda^{-b/\varepsilon}\,\mathrm{e}^{(\pi/2)\nu_k i} I_{\nu_k}\left(\frac{2\lambda a}{\varepsilon}\right)^{1/2} \\ &= \sum_{k=0}^{\infty}\left(-\frac{2a}{\varepsilon}\right)^{k+b/\varepsilon} C_k\lambda^k\,\mathrm{e}^{\alpha_k t}\sum_{l=0}^{\infty}\frac{(2a/\varepsilon)^l\lambda^l}{l!\,\Gamma(\nu_k + l + 1)} \end{aligned} \tag{18}$$

where $\alpha_k = k(b + k(\varepsilon/2))$. This is easily rewritten as follows:

$$G_x(\lambda, t) = \sum_{n=0}^{\infty}\left[(-1)^{b/\varepsilon}\left(\frac{2a}{\varepsilon}\right)^{n+b/\varepsilon} n!\sum_{k=0}^{n}\frac{(-1)^k C_k\,\mathrm{e}^{\alpha_k t}}{(n-k)!\,\Gamma(k + n + 2b/\varepsilon + 1)}\right] \tag{18'}$$

by making a transformation of the parameter n as $n=k+1$. Now using the following constraint on the generating function G_x

$$G_x(\lambda, 0)=\langle e^{-\lambda x(0)}\rangle=1$$

imposed by the initial condition of our stochastic process (3), we derive all the coefficients C_k of the expansion (18):

$$\begin{aligned}
C_0&=\left[\frac{(2/\varepsilon)^{1-b/\varepsilon}}{(-a)^{b/\varepsilon}}\right]\left[\frac{b\Gamma(2b/\varepsilon)}{0!}\right]\\
C_1&=\frac{C_0}{1!}\left(\frac{2}{\varepsilon}\right)(b+\varepsilon)\\
C_2&=\frac{C_0}{2!}\left(\frac{2}{\varepsilon}\right)(b+2\varepsilon)\left[\left(\frac{2b}{\varepsilon}+1\right)\right]\\
&\dots\\
C_k&=\frac{(2/\varepsilon)^{1-b/\varepsilon}(b+k\varepsilon)\Gamma(2b/\varepsilon+k)}{(-a)^{b/\varepsilon}k!}
\end{aligned} \tag{19}$$

and we finally obtain

$$G_x(\lambda, t)=\sum_{n=0}^{\infty}\left(\frac{2}{\varepsilon}\right)^{n+1} a^n \sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k(b+k\varepsilon)\Gamma(2b/\varepsilon+k)\, e^{\alpha_k t}}{\Gamma(k+n+2b/\varepsilon+1)}\frac{\lambda^n}{n!}. \tag{20}$$

Then simply multiplying (20) by $(-1)^{2n}$ we obtain an exact analytical expression of the moments of the process $x(t)$:

$$\langle x^n(t)\rangle=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(2a/\varepsilon)^n(2/\varepsilon)(b+k\varepsilon)\Gamma(2b/\varepsilon+k)\, e^{\alpha_k t}}{\Gamma(k+n+2b/\varepsilon+1)} \tag{21}$$

which is in agreement with theoretical results obtained using a different approach, independently by Suzuki *et al* [10] and by Brenig and Banai [11].

We note moreover that with this choice of parameters we can model the kinetics of a chemical system: the chlorite–thiosulfate reaction [12].

Case 2

We assume in the original SDE $a=0$, therefore equations (2) and (9) become

$$dx=[(b+\varepsilon/2)+\sqrt{\varepsilon}\, dW]x \tag{22}$$

$$\frac{\partial G_x(\lambda, t)}{\partial t}=\left(b+\frac{\varepsilon}{2}\right)\lambda\frac{\partial G_x}{\partial\lambda}+\frac{\varepsilon}{2}\lambda^2\frac{\partial^2 G_x}{\partial\lambda^2} \tag{23}$$

and the stochastic process is

$$x(t)=x(0)\exp(bt+\sqrt{\varepsilon}\, W(t)) \tag{24}$$

i.e. the well known linear multiplicative Gaussian Markov process.

In this case some non-trivial symmetries can arise. Again using conditions (13) and (14) we get, after some algebra, the Lie generators of the symmetries of equation (23)

$$\Lambda=B\lambda \qquad T=C \qquad G=Bg \tag{25}$$

and the subgroup invariants

$$\mu_1=g\, e^{-(B/C)t} \qquad \mu_2=\lambda\, e^{-(B/C)t} \tag{26}$$

and the similarity solution is

$$g(\lambda, t) = C_1\lambda^{\alpha} \exp\left((1-\alpha)\frac{B}{C}t\right) = C_1\lambda^{\alpha} \exp\left(\alpha\left(\alpha\frac{\varepsilon}{2}+b\right)t\right) \tag{27}$$

where

$$\alpha = \frac{(1-p)\pm\sqrt{\Delta}}{2} \qquad \Delta = (p-1)^2 - 4b \tag{28}$$

$$p = \frac{2}{\varepsilon}\left(\frac{B}{C}+\frac{\varepsilon}{2}+b\right) \qquad \frac{B}{C} = \frac{\alpha(\alpha\varepsilon/2+b)}{(1-\alpha)}.$$

With the same procedure of case 1, i.e. imposing the constraints of analyticity and of the initial condition, we obtain the generating function and the moments of the process (24)

$$G_x(\lambda, t) = \sum_n C_n\lambda^{\alpha_n} \exp(-A_n\alpha_n t) = \sum_{n=0}^{\infty} \frac{(-\lambda x(0))^n}{n!} \exp\left(\left(b+n\frac{\varepsilon}{2}\right)nt\right) = \sum_{n=0}^{\infty} \langle x^n(t)\rangle \frac{(-\lambda)^n}{n!} \tag{29}$$

which can be put in a compact analytical expression using the integral representation of the exponential and resumming, as

$$G_x(\lambda, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}k \exp\left(-\frac{k^2}{2} - \lambda x(0)\, \mathrm{e}^{bt+k\sqrt{\varepsilon t}}\right). \tag{30}$$

We want to remark finally that for a given choice of parameters of the original SDE, i.e. for a different physical model, we have a particular DEGF and a symmetry group within which other trivial and non-trivial symmetries can arise.

Moreover, we think that this procedure can also be applied to determine the generating functions associated with nonlinear stochastic processes and with passage time statistics, which is a complementary point of view to describe the stochastic dynamics of a physical system.

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